the simplest failure model is given. The authors explain that the optimization problem of the present study becomes a min-max problem. This analysis is applied to a realistic problem based on a three-ring tetrahedral truss example. So far as investigated here, an actuator location with high fault tolerance for static shape control can be obtained by an optimization with actuator failure consideration.

Acknowledgments

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Zigzag Theory for Composite Laminates

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Fundamentals

HIGH-ORDER shear deformation theories (HSDT) have been widely used in composite laminate analysis. Lo et al.[1] have used unified notations to document the development of HSDT according to the order of thickness coordinate z of the assumed inplane displacements. As a result, they have also presented a third-order shear deformation theory for composite laminate analysis, i.e.,

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + u_1(x,y)z + u_2(x,y)z^2 + u_3(x,y)z^3 \\ v(x,y,z) &= v_0(x,y) + v_1(x,y)z + v_2(x,y)z^2 + v_3(x,y)z^3 \\ w(x,y,z) &= w_0(x,y) + w_1(x,y)z + w_2(x,y)z^2 \end{aligned} \quad (1)$$

The predictions of in-plane stresses from HSDT are reasonably good. However, with the assumptions of continuous functions of u and v through the laminate thickness, the kinky, zigzag distribution of in-plane displacements cannot be obtained from HSDT. To remove the deficiency and to improve the accuracy of transverse stress prediction, layerwise theories (LT), which are based on assumed displacements for individual layers, have been proved to be very promising techniques. A third-order layerwise theory has been presented by Lu and Liu[2] and can be expressed by the unified notations as used in Eqs. (1),

$$\begin{aligned} u^k(x,y,z) &= u_0^k(x,y) + u_1^k(x,y)z + u_2^k(x,y)z^2 + u_3^k(x,y)z^3 \\ v^k(x,y,z) &= v_0^k(x,y) + v_1^k(x,y)z + v_2^k(x,y)z^2 + v_3^k(x,y)z^3 \\ w(x,y,z) &= w_0(x,y) + w_1(x,y)z + w_2(x,y)z^2 \end{aligned} \quad (2)$$

where the superscript k represents the layer order.

Although the layerwise theory gives excellent displacements and stresses (both in-plane and transverse), it suffers from a major computational drawback. The total number of degrees of freedom is dependent on the number of composite layers. As the layer number increases, the computational effort becomes very demanding. As for compromising techniques to HSDT and LT, a few zigzag theories (ZT) based on different displacement fields have been developed, e.g., Refs. 3–7. This study aims at presenting a zigzag theory by utilizing the same notations as those used in Eqs. (2).

By following Eqs. (2), the in-plane displacements for each composite layer are assumed to consist of up to third-order terms. To have the total number of degrees of freedom independent of layer number, only two coefficients of u_i are allowed to be layer dependent. The remaining two coefficients are layer-independent variables. Since the characteristic of the zigzag theories is that only their zeroth-order and first-order terms are designated as layer dependent, the displacements of the kth layer of a zigzag theory can be defined as follows:

$$\begin{aligned} u^k(x,y,z) &= u_0^k(x,y) + u_1^k(x,y)z + u_2(x,y)z^2 + u_3(x,y)z^3 \\ w^k(x,y,z) &= w_0(x,y) + w_1(x,y)z + w_2(x,y)z^2 \end{aligned} \quad (3)$$

It should be noted that Eqs. (3) are for a two-dimensional, third-order theory. A more generalized theory that contains high-order terms can certainly be established. In addition, it is worthwhile to point out that the coefficients of the in-plane displacement polynomial are variables to be determined by variational process and continuity conditions on the laminate interfaces instead of special functions as defined in other zigzag theories. Details of the coordinates, layer order, and interface locations can be found in Fig. 1.

In this study, linear strain-displacement relations are employed:

$$\varepsilon_x^k = \frac{\partial u^k}{\partial x} \qquad \varepsilon_z^k = \frac{\partial w^k}{\partial z} \qquad \gamma_{xz}^k = \frac{\partial u^k}{\partial z} + \frac{\partial w^k}{\partial x} \quad (4)$$

Since the primary objective of this study is to evaluate the proposed zigzag theory, a closed-form solution that is free from numerical error is highly desired. As a consequence, only cross-ply laminates are considered. (For other composite laminations, numerical analysis such as finite element method is required.) The constitutive equations for the kth layer are given by

$$\begin{Bmatrix} \sigma_x^k \\ \sigma_z^k \\ \tau_{xy}^k \end{Bmatrix} = \begin{bmatrix} Q_{11}^k & Q_{13}^k & 0 \\ Q_{13}^k & Q_{33}^k & 0 \\ 0 & 0 & Q_{55}^k \end{bmatrix} \begin{bmatrix} \varepsilon_x^k \\ \varepsilon_z^k \\ \gamma_{xz}^k \end{bmatrix} \quad (5)$$

where Q_{ij}^k are stiffness components.

Formulation

Based on Eqs. (3), the number of layer-dependent variables in an n-layer composite laminate is $2n$. These variables can be replaced by layer-independent variables through the enforcement of continuity conditions of both displacement and transverse shear stress on the laminate interfaces. If the composite laminate has perfect bonding on the interfaces, the following continuity conditions should be satisfied:

$$\begin{aligned} u^{k-1}\big|_{z=z_k} &= u^k\big|_{z=z_k} \\ \tau_{xz}^{k-1}\big|_{z=z_k} &= \tau_{xz}^k\big|_{z=z_k} \qquad k = 2,3,4,\ldots,n \end{aligned} \quad (6)$$

By substituting Eqs. (3–5) into Eqs. (6), u_0^k and u_1^k are no longer layer-dependent variables. They can be replaced by two new layer-independent variables u_0 and u_1. And the coefficients of the polynomial equation are functions of layer properties and coordinates, i.e.,

$$\begin{aligned} u^k &= u_0 + \left(R_1^k + A_1^k z\right)u_1 + \left(R_2^k + A_2^k z + z^2\right)u_2 \\ &\quad + \left(R_3^k + A_3^k z + z^3\right)u_3 + \left(R_4^k + A_4^k z\right)w_{0,x} \\ &\quad + \left(R_5^k + A_5^k z\right)w_{1,x} + \left(R_6^k + A_6^k z\right)w_{2,x} \end{aligned} \quad (7)$$

$$w^k(x,y,z) = w_0(x,y) + w_1(x,y)z + w_2(x,y)z^2$$

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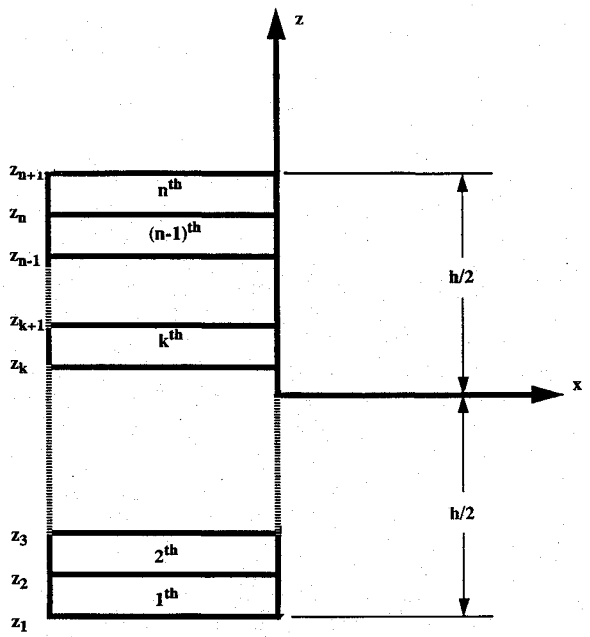

Fig. 1 Coordinate system, layer order, and interface locations.

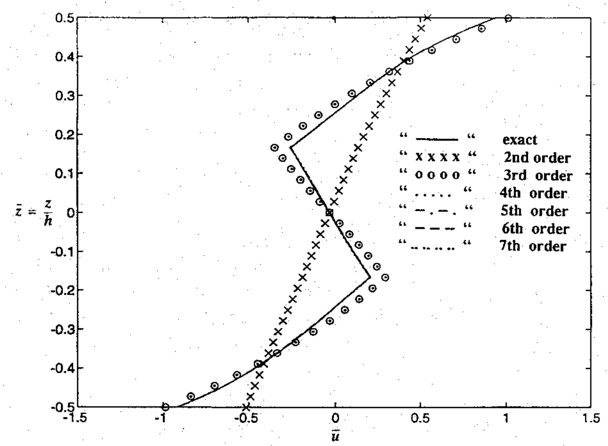

Fig. 2 Normalized in-plane displacement $\bar{u}$ at the edge of a [0/90/0] laminate.

where

$$A_1^k = \frac{Q_{55}^1}{Q_{55}^k} \qquad A_2^k = \frac{2}{Q_{55}^k}\sum_{j=2}^{k}\alpha_j z_j \qquad A_3^k = \frac{3}{Q_{55}^k}\sum_{j=2}^{k}\alpha_j z_j^2$$

$$A_4^k = A_1^k - 1 \qquad A_5^k = \tfrac{1}{2}A_2^k \qquad A_6^k = \tfrac{1}{3}A_3^k$$

$$R_i^k = \sum_{l=2}^{k}\left(A_i^{l-1} - A_i^l\right) z_l$$

and $\alpha_k = Q_{55}^{k-1} - Q_{55}^k$.

The total degrees of freedom of the zigzag theory are now only dependent on the five layer-independent variables that can be further reduced by applying free shear traction conditions on both surfaces of the laminate. The enforcement of the boundary conditions is a common exercise in laminate analysis, i.e.,

$$\tau_{xz}\left(\pm\frac{h}{2}\right) = 0 \tag{8}$$

where h is the total thickness of the composite laminate. As a consequence, the number of the total degrees of freedom of the zigzag theory become three.

To verify the present theory for laminated plate analysis, the plane-strain problem studied by Pagano[8] is investigated. A composite laminate of infinitely long strip is simply supported along two

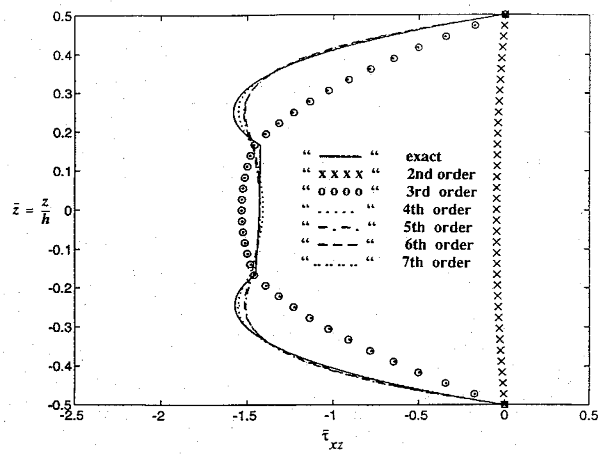

Fig. 3 Normalized transverse shear stresses $\bar{\tau}_{xz}$ at the edge of a [0/90/0] laminate.

longer edges and subjected to a sinusoidal tension on the top surface, i.e., $q(x, h/2) = q_0 \sin(\pi x/l)$ where l is the span between the supported edges. By virtue of the principle of virtual displacement as shown next:

$$\int_\Omega \left\{ \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left[\begin{pmatrix} \sigma_x^k \\ \sigma_z^k \\ \tau_{xz}^k \end{pmatrix}^T \begin{pmatrix} \delta\varepsilon_x^k \\ \delta\varepsilon_z^k \\ \delta\gamma_{xz}^k \end{pmatrix} \right] \mathrm{d}z - q(\delta w) \right\} \mathrm{d}x\,\mathrm{d}y = 0 \tag{9}$$

it is possible to obtain closed-form solutions for cross-ply laminates. Once the independent variables are identified, the displacements and stresses of the composite laminate can then be calculated.

Influence of High-Order Terms

In addition to the previous formulation, it is interesting to investigate the influence of the order of the layer-independent terms on the accuracy of displacement and stress predictions. In this study, layer-independent terms of up to seventh order are added to Eqs. (3) without too much effort, i.e.,

$$u^k = u_0^k + u_1^k z + u_2 z^2 + u_3 z^3 + u_4 z^4 + u_5 z^5 + u_6 z^6 + u_7 z^7$$
$$w^k = w_0 + w_1 z + w_2 z^2 \tag{10}$$

In numerical analysis, the same material properties utilized by Pagano are also investigated in this study. A composite laminate of [0/90/0] with aspect ratio $l/h = 4$ is examined due to its common use in assessing various laminate theories (HSDT, LT, and ZT). Figure 2 shows the normalized in-plane displacement $\bar{u}$. The kinky, zigzag distribution can be perfectly represented by the generalized zigzag theory with layer-independent terms of third order and above. As can be seen from Fig. 3, the exact values of $\bar{\tau_{xz}}$ are a little more difficult to obtain. Although the fifth order and above have significant effect on the transverse shear stress distribution, they fail to give distinct kinks on the laminate interfaces. It is believed that this deficiency cannot be removed by simply increasing the order of the layer-independent terms.

As compared with the high-order shear deformation theories,[1] the zigzag theory gives kinky in-plane displacement and continuous interlaminar shear stress. As compared with the layerwise theory,[2] the zigzag theory is layer independent. As a consequence, the zigzag theory is a compromising theory between the shear deformation theory and the layerwise theory and holds both advantages of numerical accuracy and computational efficiency.

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Singular Optima in Geometrical Optimization of Structures

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Introduction

A POINT is said to be a local (relative) minimum if it has the least function value in its neighborhood but not necessarily the least function value for all the feasible region. In this Note, a point is said to be a singular optimum if it has the least function value in its neighborhood, but this neighborhood is a reduced (degenerate) feasible region, formed by assuming certain variables as zero. Singular optima are usually associated with changes in the topology of the structure. If the optimal solution is a singular point in the design space, it might be difficult or even impossible to arrive at the true optimum by numerical search algorithms. The singularity of the optimal topology in cross-sectional optimization of truss structures was first shown by Sved and Ginos.[1] Singular optima of grillages[2,3] and some properties of singular optimal topologies[4,5] were studied later. In particular, problems of continuous constraint functions have been discussed, and limiting stresses obtained in cases of elimination of members have been defined.[5]

In this Note, the effect of some preassigned parameters on singular optima in cross-sectional optimization is first demonstrated. Then, singular optima that might occur in geometrical optimization are presented. In simultaneous optimization of geometrical and cross-sectional variables, it is possible that some joints tend to coalesce during the solution process. It will be shown that under certain circumstances, reduced optimal structures obtained by elimination of members that coincide with others due to coalescence of joints, might represent singular optima that cannot be reached by simple numerical optimization.

Effect of Geometrical Preassigned Parameters

The following notation will be used for the various optimum design points: G is the global optimum, L local optimum, GS global singular optimum, and LS local singular optimum. To illustrate singular and local optima in the optimization of cross-sectional variables, consider the three beam grillage shown in Fig. 1 and subjected to two concentrated loads $P = 1$ at the intersections (arbitrary dimensions are assumed in all examples). Assume constant depth of 2.45 and the width of the cross sections for the longitudinal and transverse beams, denoted as X_1 and X_2, respectively, as design variables. The allowable stress is $\sigma^U = 10$, and the member's lengths are $\ell_x = 10.0$ and $\ell_y = 14.0$. Neglecting torsional rigidity of the elements, assuming only stress constraints and the volume of material as an objective function, the design space is shown in Fig. 2a and the solution process converges to point A. However, since $X_2 = 0$ at this point, the two transverse members and the constraint $\sigma_2 \leq \sigma^U$ are eliminated. The resulting global singular optimum is at point $GS(X_1 = 1.0, X_2 = 0, Z = 73.5)$ and not at point $A(X_1 = 1.275, X_2 = 0, Z = 93.7)$. It can be observed that the line segment A–GS is a reduced (degenerate) part of the feasible region. In addition, both points A and GS represent a topology of only the longitudinal beam. Although the optimum is a singular point in the design space, there is a single feasible region. Two or more separated regions might exist if upper or lower limits on design variables are considered.[5]

It is instructive to note that modifications in the objective function coefficients might result in different points of convergence and optima. Assuming, for example, the objective function

$$Z = 2.45(TX_1 + 56X_2) \rightarrow \min \tag{1}$$

where T is a predetermined coefficient, then the various possible solutions are summarized in Table 1. It can be seen that convergence to the true optimum is achieved only if $T \geq 39.2$. It should be emphasized that singular optima exist also in alternative formulations of the problem,[6] such as simultaneous analysis and design (SAND). However, such optima do not exist in optimal plastic design, where the compatibility conditions are neglected.[6]

It has been noted[6] that changes in various preassigned parameters (such as geometrical parameters, allowable stresses, and limits on design variables and loadings) might result in singular optima. To illustrate the effect of geometrical parameters, assume various geometries for the grillage of Fig. 1. Results for $\ell_y = 14.0$, $\ell_y = 60.0$, and $\ell_y = 6.0$ are shown in Fig. 2 and in Table 2. As noted earlier, for $\ell_y = 14.0$ (Fig. 2a) the singular global optimum is at point GS. For $\ell_y = 60.0$ (Fig. 2b) the optimum is at point G, and for $\ell_y = 6.0$ (Fig. 2c) the optimum is at point G, both being nonsingular global optima.

Table 1 Effect of objective function coefficients, grillage, Fig. 1

Value of T	Point of convergence	True optimum
$T < 30.75$	A	GS
$T = 30.75$	Along A–B	GS
$30.75 < T < 39.2$	B	GS
$T = 39.2$	B	B and GS
$T > 39.2$	B	B

Table 2 Effect of geometry, grillage, Fig. 1

Figure	ℓ_y	Point	X_1	X_2	Z
2a	14	A	1.275	0	93.7
		GS	1.0	0	73.5
2b	60	G	1.0	0	73.5
		B	0	43.2	25396
2c	6	A	69.4	0	5103
		G	0	3.0	176.4

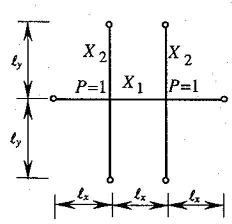

Fig. 1 Grillage.

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